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Introduction

1.1 Exact Controllability

What is the exact controllability? Let us begin from the simplest situation. Consider the following system of linear ODEs

$$\frac{dX}{dt} = AX + Bu, \tag{1.1}$$

where t is the independent variable (time), $X = (X_1, \cdots, X_N)$ is the state variable, $u = (u_1, \cdots, u_m)$ is the control variable, A and B are $N \times N$ and $N \times m$ constant matrices respectively.

This system possesses the **exact controllability** on the interval $[0, T]$ ($T > 0$), if, for any given initial data X_0 at $t = 0$ and any given final data X_T at $t = T$, we can find a control function $u = u(t)$ on $[0, T]$, such that the solution $X = X(t)$ to the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt} = AX + Bu(t), & (1.2) \\ t = 0: \quad X = X_0 & (1.3) \end{cases}$$

verifies exactly the final condition

$$t = T: \quad X = X_T. \tag{1.4}$$

It is well-known that system (1.1) possesses the exact controllability on $[0, T]$, if and only if the matrix

$$[B \vdots AB \vdots \cdots \vdots A^{N-1}B] \tag{1.5}$$

is full-rank (cf. [77]). Hence, if system (1.1) is exactly controllable on an interval $[0, T]$ ($T > 0$), then it is also exactly controllable on any interval $[0, T_1]$ ($T_1 > 0$), in particular, the exact controllability can be realized almost immediately.

We now consider the exact controllability for hyperbolic systems of PDEs. For this purpose, several points different from the ODE case should be pointed out as follows.

1. In order to solve a hyperbolic system on a bounded domain (or on a domain with boundary), one should prescribe suitable boundary conditions. As a result, the control may be an **internal control** appearing in the equation like in the ODE case and acting on the whole domain or a part of domain, or a **boundary control** appearing in the boundary conditions and acting on the whole boundary or a part of boundary.

Since the boundary control is much easier than the internal control to be handled in practice, we concentrate our attention mainly on the **exact boundary controllability**, namely, the exact controllability realized only by boundary controls.

The **exact boundary controllability** means that there exists $T > 0$ such that by means of boundary controls, the system (hyperbolic equations together with boundary conditions) can drive any given initial data at $t = 0$ to any given final data at $t = T$.

If the exact boundary controllability can be realized only for small (in some sense!) initial data and final data, it is called to be a **local exact boundary controllability**; otherwise, a **global exact boundary controllability**.

2. Since the hyperbolic wave has a finite speed of propagation, the exact boundary controllability time $T > 0$ should be suitably large.

In fact, for any given initial data, by solving the corresponding **forward** Cauchy problem, there is a unique solution on its maximum determinate domain.

Similarly, for any given final data, by solving the corresponding **backward** Cauchy problem, there is a unique solution on its maximum determinate domain.

In order to ensure the consistency, these two maximum determinate domains should not intersect each other (Figure 1.1), then $T(> 0)$ must be suitably large.

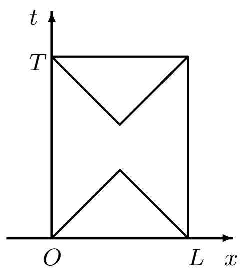

Figure 1.1

On the other hand, from the point of view of applications, $T(>0)$ should be chosen as small as possible.

3. For the weak solution to quasilinear hyperbolic systems, which includes shock waves and corresponds to an irreversible process, generically speaking, it is impossible to have the exact boundary controllability for any arbitrarily given initial and final states (cf. [7]). Of course, by requiring certain additional restrictions on the initial state and the final state (particularly on the later) and, perhaps, suitably weakening the definition of controllability, it is still possible to consider the exact boundary controllability in the framework of weak solutions, however, up to now several results obtained in this direction with different methods are only for very special quasilinear hyperbolic systems (the scalar convex conservation law [3–4], [32], genuinely nonlinear systems of Temple class [2] and the p-system in isentropic gas dynamics [17]). Hence, in order to give a general and systematic presentation, in this book we restrict ourselves to the consideration in the framework of classical solutions, namely, the solution under consideration to the hyperbolic system means its classical solution which corresponds to a reversible process.

We know that for nonlinear hyperbolic problems, there is always the local existence and uniqueness of classical solutions, provided that the initial data and the boundary data are smooth and suitable conditions of compatibility hold; but, generically speaking, the classical solution exists only locally in time (see [33–34], [39], [41]). However, as we said before, in order to guarantee the exact boundary controllability, we should have a classical solution on the interval $[0, T]$, where $T > 0$ might be suitably large. This kind of classical solution is called to be a **semi-global classical solution** (see Chapter 2), which is different from either the local classical solution or the global classical solution (cf. [10], [53], [60–62]).

Thus, the existence of semi-global classical solution is an important basis for the exact boundary controllability.

Since, generically speaking, the semi-global classical solution to quasilinear hyperbolic systems exists only for small initial and boundary data and keeps small in its existence domain (see Chapter 2), in general one can only expect to have the local exact controllability in the quasilinear case. However, it is still possible to get the global exact controllability in some special cases (see Remark 3.9).

In the case of hyperbolic PDEs, most studies on the controllability are concentrated on the wave equation

$$u_{tt} - \triangle u = 0 \tag{1.6}$$

(cf. [73–75] and the references therein). Moreover, there are some results for semilinear wave equations

$$u_{tt} - \triangle u = F(u) \tag{1.7}$$

(cf. [15–16], [35], [99–100], [103]). However, in the quasilinear case, very few results have been published even for the 1-D quasilinear hyperbolic PDEs (see [9]).

In this book we shall consider the exact boundary controllability for first order quasilinear hyperbolic systems with general nonlinear boundary conditions in one-space dimensional case.

More precisely, we consider the following first order 1-D quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.8}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u)$ is an $n \times n$ matrix with smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a smooth vector function of u with

$$F(0) = 0. \tag{1.9}$$

Obviously, $u = 0$ is an **equilibrium** of (1.8).

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ $(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u). \tag{1.10}$$

In particular, when $A(u)$ has n distinct real eigenvalues

$$\lambda_i(u) < \lambda_2(u) < \cdots < \lambda_n(u) \tag{1.11}$$

on the domain under consideration, system (1.8) is called to be **strictly hyperbolic**.

Suppose that there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m;\ s = m + 1, \cdots, n). \tag{1.12}$$

In this situation, the subscripts $r = 1, \cdots, m$ (resp. $s = m + 1, \cdots, n$) are always used to correspond to the negative (resp. positive) eigenvalues.

Let

$$v_i = l_i(u)u \qquad (i = 1, \cdots, n). \tag{1.13}$$

v_i is called to be the **diagonal variable** corresponding to the i-th eigenvalue $\lambda_i(u)$.

The boundary conditions are given by

$$x = 0: \quad v_s = G_s(t, v_1, \cdots, v_m) + H_s(t) \tag{1.14}$$
$$(s = m + 1, \cdots, n),$$
$$x = L: \quad v_r = G_r(t, v_{m+1}, \cdots, v_n) + H_r(t) \tag{1.15}$$
$$(r = 1, \cdots, m),$$

where L is the length of the space interval $0 \le x \le L$, G_i $(i = 1, \cdots, n)$ are suitably smooth functions and, without loss of generality, we assume

$$G_i(t, 0, \cdots, 0) \equiv 0 \qquad (i = 1, \cdots, n); \tag{1.16}$$

moreover, all $H_i(t)(i = 1, \cdots, n)$ or a part of $H_i(t)(i = 1, \cdots, n)$ will be chosen as boundary controls.

(1.14)–(1.15) are the most general nonlinear boundary conditions to guarantee the well-posedness for the forward problem, the characters of which can be shown as

1) The number of boundary conditions on $x = 0$ (resp. on $x = L$) is equal to the number of positive (resp. negative) eigenvalues.

2) The boundary conditions on $x = 0$ (resp. on $x = L$) are written in the form that the diagonal variables v_s $(s = m+1, \cdots, n)$ corresponding to positive eigenvalues (resp. the diagonal variables v_r $(r = 1, \cdots, m)$ corresponding to negative eigenvalues) are explicitly expressed by the other diagonal variables.

For any given initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L \tag{1.17}$$

and any given final condition

$$t = T: \quad u = \Phi(x), \quad 0 \le x \le L \tag{1.18}$$

with small C^1 norms $\|\varphi\|_{C^1[0,L]}$ and $\|\Phi\|_{C^1[0,L]}$, by means of the theory on the semi-global classical solution (see Chapter 2), we shall present a direct and simple constructive method to show the following theorems on the local exact boundary controllability (see Chapter 3).

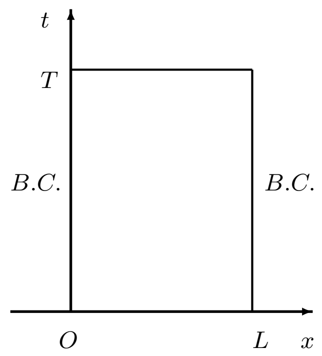

Figure 1.2

Theorem 1.1 (Two-sided control, [55]). *If*

$$T > L \max_{\substack{r=1,\cdots,m\\ s=m+1,\cdots,n}} \Big(\frac{1}{|\lambda_r(0)|}, \frac{1}{\lambda_s(0)}\Big), \tag{1.19}$$

then there exist boundary controls $H_i(t)$ $(i = 1, \cdots, n)$ *with small* $C^1[0,T]$ *norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global* C^1 *solution* $u = u(t,x)$ *with small* C^1 *norm on the domain* $R(T) = \{(t,x)|\ 0 \le t \le T, \quad 0 \le x \le L\}$, *which verifies exactly the final condition (1.18)* (Figure 1.2).

Theorem 1.2 (One-sided control, [54]). *Suppose that the number of positive eigenvalues is not bigger than that of negative ones:*

$$\bar{m} \overset{\text{def}}{=} n - m \le m, \quad i.e.,\ n \le 2m. \tag{1.20}$$

Suppose furthermore that boundary condition (1.14) on $x = 0$ *can be equivalently rewritten in a neighbourhood of* $u = 0$ *as*

$$x = 0: \quad v_{\bar{r}} = \bar{G}_{\bar{r}}(t, v_{\bar{m}+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + \bar{H}_{\bar{r}}(t) \quad (\bar{r} = 1, \cdots, \bar{m}), \tag{1.21}$$

where

$$\bar{G}_{\bar{r}}(t, 0, \cdots, 0) \equiv 0 \qquad (\bar{r} = 1, \cdots, \bar{m}), \tag{1.22}$$

then

$$\begin{aligned} &\|\bar{H}_{\bar{r}}\|_{C^1[0,T]}\ (\bar{r} = 1, \cdots, \bar{m})\ \textit{small enough} \\ \Longleftrightarrow &\|H_s\|_{C^1[0,T]}\ (s = m+1, \cdots, n)\ \textit{small enough.} \end{aligned} \tag{1.23}$$

If

$$T > L\Big(\max_{r=1,\cdots,m} \frac{1}{|\lambda_r(0)|} + \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(0)}\Big), \tag{1.24}$$

then, for any given $H_s(t)$ $(s = m+1, \cdots, n)$ *with small* $C^1[0,T]$ *norm, satisfying the conditions of* C^1 *compatibility at the points* $(t,x) = (0,0)$ *and*

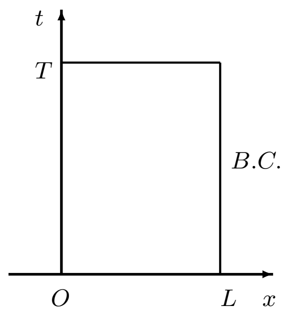

Figure 1.3

$(T, 0)$ respectively, there exist boundary controls $H_r(t)$ $(r = 1, \cdots, m)$ at $x = L$ with small $C^1[0, T]$ norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T) = \{(t, x) | \ 0 \le t \le T, \quad 0 \le x \le L\}$, which verifies exactly the final condition (1.18) (Figure 1.3).

Theorem 1.3 (Two-sided control with less controls, [96]). *Suppose that the number of positive eigenvalues is less than that of negative ones:*

$$\bar{m} \overset{\text{def}}{=} n - m < m, \quad i.e., \ n < 2m. \tag{1.25}$$

Suppose furthermore that, without loss of generality, the first $\bar{m}$ boundary conditions in (1.15) at $x = L$, namely,

$$x = L: \quad v_{\bar{r}} = G_{\bar{r}}(t, v_{m+1}, \cdots, v_n) + H_{\bar{r}}(t) \quad (\bar{r} = 1, \cdots, \bar{m}), \tag{1.26}$$

can be equivalently rewritten in a neighbourhood of $u = 0$ as

$$x = L: \quad v_s = \bar{G}_s(t, v_1, \cdots, v_{\bar{m}}) + \bar{H}_s(t) \quad (s = m + 1, \cdots, n), \tag{1.27}$$

where

$$\bar{G}_s(t, 0, \cdots, 0) \equiv 0 \quad (s = m + 1, \cdots, n), \tag{1.28}$$

then

$$\begin{aligned} &\|\bar{H}_s\|_{C^1[0,T]} \ (s = m + 1, \cdots, n) \ \textit{small enough} \\ \Longleftrightarrow &\|H_{\bar{r}}\|_{C^1[0,T]} \ (\bar{r} = 1, \cdots, \bar{m}) \ \textit{small enough.} \end{aligned} \tag{1.29}$$

If $T > 0$ satisfies (1.24), then, for any given $H_{\bar{r}}(t)$ $(\bar{r} = 1, \cdots, \bar{m})$ with small $C^1[0, T]$ norm, satisfying the conditions of C^1 compatibility at the points $(t, x) = (0, L)$ and (T, L) respectively, there exist boundary controls $H_s(t)$ $(s = m + 1, \cdots, n)$ at $x = 0$ and $H_{\bar{\bar{r}}}(t)(\bar{\bar{r}} = \bar{m} + 1, \cdots, m)$ at $x = L$ with small $C^1[0, T]$ norm, such that the corresponding mixed initial-boundary value problem (1.8), (1.17) and (1.14)–(1.15) admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T) = \{(t, x) \big| \ 0 \le t \le T, \ 0 \le x \le L\}$, which verifies exactly the final condition (1.18) (Figure 1.4).

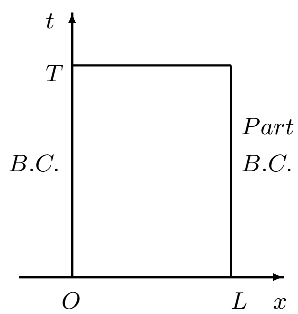

Figure 1.4

Remark 1.1. In the case of two-sided control, the number of boundary controls is equal to n, the number of unknown variables, namely, that of all the eigenvalues.

Remark 1.2. In the case of one-sided control, the number of boundary controls is reduced to the maximum value between the number of positive eigenvalues and the number of negative eigenvalues, and the boundary controls act only on the side with more boundary conditions, however, the controllability time must be enlarged.

In particular, when the number of positive eigenvalues is equal to the number of negative eigenvalues, boundary controls can act on each side.

Remark 1.3. In the case of two-sided control with less controls, both the number of boundary controls and the controllability time are as in the case of one-sided control, however, one needs all the boundary controls acting on the side with less boundary conditions and a part of boundary controls acting on the side with more boundary conditions.

Remark 1.4. The estimate on the exact controllability time T in Theorems 1.1–1.3 is sharp.

Remark 1.5. The boundary controls which realize the exact boundary controllability are not unique.

1.2 Exact Observability

Consider the system of linear ODEs

$$\frac{dX}{dt} = AX, \tag{1.30}$$

where $X = (X_1, \cdots, X_N)$ and A is an $N \times N$ constant matrix.

For any given initial data

$$t = 0: \quad X = X_0, \tag{1.31}$$

Cauchy problem (1.30)–(1.31) admits a unique solution $X = X(t)$.

Let

$$Y(t) = DX(t) \tag{1.32}$$

be the corresponding **observed value**, where D is an $m \times N$ constant matrix.

System (1.30) with (1.32) possesses the **exact observability** on the interval $[0, T]$ $(T > 0)$, if the observed value $Y(t)$ on the interval $[0, T]$ determines uniquely the initial data X_0 (then the solution $X(t)$ on any interval $[0, \tilde{T}]$).

It is well-known that system (1.30) with (1.32) possesses the exact observability on $[0, T]$, if and only if the matrix

$$\begin{bmatrix} D \\ DA \\ \vdots \\ DA^{N-1} \end{bmatrix} \tag{1.33}$$

is full-rank (cf. [77]). Hence, the exact observability on an interval $[0, T]$ ($T > 0$) implies the exact observability on any given interval $[0, T_1]$ ($T_1 > 0$), then the exact observability can be realized almost immediately.

System (1.30) with (1.32) possesses the exact observability on $[0, T]$, if and only if

$$Y(t) \equiv 0, \quad \forall\, t \in [0, T] \tag{1.34}$$

implies

$$X_0 = 0. \tag{1.35}$$

A **strengthened form of exact observability** is as follows: System (1.30) with (1.32) possesses the exact observability on the interval $[0, T]$ ($T > 0$), if the following observability inequality holds:

$$|X_0| \leq C\|Y\|, \tag{1.36}$$

where C is a positive constant independent of $Y(t)$ but possibly depending on T, $\|Y\|$ is a suitable norm of $Y(t)$ on $[0, T]$.

Remark 1.6. The usual inequality

$$\|Y\| \leq C|X_0| \tag{1.37}$$

coming from the well-posedness is a direct inequality, while, the observability inequality (1.36) is an inverse inequality.

In what follows, we always assume that the observed value is accurate, i.e., there is no measuring error in the observation.

In the case of hyperbolic PDEs, the observed value may be the **boundary observed value** (on the whole boundary or on a part of boundary) or the **internal observed value** (on the whole domain or on a part of domain).

For the same reason as in the controllability case, we concentrate our attention mainly on the **exact boundary observability**, namely, the exact observability realized only by the boundary observation.

We have still the **local exact boundary observability** or the **global exact boundary observability**.

As in the controllability case, the exact boundary observability time $T > 0$ should be suitably large, then we still need the existence and uniqueness of the semi-global classical solution as a necessary basis. On the other hand, for the purpose of applications, we need to take $T > 0$ as small as possible.

Consider the quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u) \tag{1.8}$$

with

$$F(0) = 0. \tag{1.9}$$

Suppose that there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m;\ s = m+1, \cdots, n). \tag{1.12}$$

Let

$$v_i = l_i(u)u \quad (i = 1, \cdots, n) \tag{1.13}$$

be the inner product between the i-th left eigenvector $l_i(u)$ and u. The boundary conditions are given as

$$x = 0 : v_s = G_s(t, v_1, \cdots, v_m) \quad (s = m+1, \cdots, n), \tag{1.38}$$

$$x = L : v_r = G_r(t, v_{m+1}, \cdots, v_n) \quad (r = 1, \cdots, m), \tag{1.39}$$

where G_i $(i = 1, \cdots, n)$are suitably smooth and

$$G_i(t, 0, \cdots, 0) \equiv 0 \quad (i = 1, \cdots, n). \tag{1.16}$$

$u = 0$ is an **equilibrium** to system (1.8) with (1.38)–(1.39).

For getting the exact boundary observability, the essential principle of choosing the observed value on the boundary is that the observed value together with the boundary conditions can uniquely determine the value $u = (u_1, \cdots, u_n)$ on the boundary.

Following this principle, the observed value at $x = 0$ should be essentially the diagonal variables $v_r = \bar{v}_r(t)$ $(r = 1, \cdots, m)$ corresponding to the negative eigenvalues, then, by means of the boundary condition (1.38), we get

$$v_s = \bar{v}_s(t) \stackrel{\text{def}}{=} G_s(t, \bar{v}_1(t), \cdots, \bar{v}_m(t)) \quad (s = m+1, \cdots, n) \tag{1.40}$$

and then $u = \bar{u}(t)$ at $x = 0$.

Similarly, the observed value at $x = L$ should be essentially $v_s = \bar{\bar{v}}_s(t)$ $(s = m+1, \cdots, n)$ corresponding to the positive eigenvalues, then, by means of the boundary (1.39), we get

$$v_r = \bar{\bar{v}}_r(t) \stackrel{\text{def}}{=} G_r(t, \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) \quad (r = 1, \cdots, m) \tag{1.41}$$

and then $u = \bar{\bar{u}}(t)$ at $x = L$.

By means of the theory on the semi-global classical solution, a direct and simple constructive method is presented in Chapter 4 to give the following theorems on the local exact boundary observability (see [49], [51]).

In this constructive way, the observability inequality as an inverse inequality becomes a direct consequence of several direct inequalities obtained by solving some well-posed problems.

Theorem 1.4 (Two-sided observation). *If*

$$T > L \max_{\substack{r=1,\cdots,m \\ s=m+1,\cdots,n}} \Big(\frac{1}{|\lambda_r(0)|}, \frac{1}{\lambda_s(0)}\Big), \tag{1.19}$$

then, for any given initial data

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L \tag{1.42}$$

with small C^1 norm $\|\varphi\|_{C^1[0,L]}$, satisfying the conditions of C^1 compatibility at the points $(t,x) = (0,0)$ and $(0,L)$ respectively, by means of the observed values $v_r = \bar{v}_r(t)$ $(r = 1,\cdots,m)$ corresponding to the negative eigenvalues at $x = 0$ and the observed values $v_s = \bar{\bar{v}}_s(t)$ $(s = m+1,\cdots,n)$ corresponding to the positive eigenvalues at $x = L$ on the interval $[0,T]$, we can uniquely determine the initial data $\varphi(x)$ and have the following observability inequality

$$\|\varphi\|_{C^1[0,L]} \le C\big(\sum_{r=1}^{m} \|\bar{v}_r\|_{C^1[0,T]} + \sum_{s=m+1}^{n} \|\bar{\bar{v}}_s\|_{C^1[0,T]}\big), \tag{1.43}$$

where C is a positive constant.

Theorem 1.5 (One-sided observation). *Suppose that the number of positive eigenvalues is not bigger than the number of negative eigenvalues:*

$$\bar{m} \stackrel{\text{def}}{=} n - m \le m, \quad i.e., \ n \le 2m. \tag{1.20}$$

Suppose furthermore that in a neighborhood of $u = 0$, the boundary condition (1.39) at $x = L$ implies

$$x = L: \quad v_s = \bar{G}_s(t, v_1, \cdots, v_{\bar{m}}, v_{\bar{m}+1}, \cdots, v_m) \\ (s = m+1, \cdots, n) \tag{1.44}$$

with

$$\bar{G}_s(t, 0, \cdots, 0) \equiv 0 \quad (s = m+1, \cdots, n). \tag{1.45}$$

Let

$$T > L\Big(\max_{r=1,\cdots,m} \frac{1}{|\lambda_r(0)|} + \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(0)}\Big). \tag{1.24}$$

For any given initial data $\varphi(x)$ with small $C^1[0,L]$ norm, satisfying the conditions of C^1 compatibility at the points $(t,x) = (0,0)$ and $(0,L)$ respectively, by means of the observed values $v_r = \bar{v}_r(t)$ $(r = 1,\cdots,m)$ corresponding to the negative eigenvalues at $x = 0$ on the interval $[0,T]$, we can uniquely determine the initial data $\varphi(x)$ and have the following observability inequality

$$\|\varphi\|_{C^1[0,L]} \le C \sum_{r=1}^{m} \|\bar{v}_r\|_{C^1[0,T]}, \tag{1.46}$$

where C is a positive constant.

Theorem 1.6 (Two-sided observation with less observed values). *Suppose that the number of positive eigenvalues is less than that of negatives ones:*

$$\bar{m} \stackrel{\text{def}}{=} n - m < m, \quad i.e., \ n < 2m. \tag{1.25}$$

Suppose furthermore that in a neighbourhood of $u = 0$, the boundary condition (1.38) at $x = 0$ can be equivalently rewritten as

$$x = 0: \quad v_{\bar{r}} = \bar{G}_{\bar{r}}(t, v_{\bar{m}+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) \quad (\bar{r} = 1, \cdots, \bar{m}) \tag{1.47}$$

with

$$\bar{G}_{\bar{r}}(t, 0, \cdots, 0) \equiv 0 \quad (\bar{r} = 1, \cdots, \bar{m}). \tag{1.48}$$

Let $T > 0$ satisfy (1.24). For any given initial data $\varphi(x)$ with small $C^1[0, L]$ norm, satisfying the conditions of C^1 compatibility at the points $(t, x) = (0, 0)$ and $(0, L)$ respectively, by means of the observed values $v_{\bar{s}} = \bar{v}_{\bar{s}}(t)$ $(\bar{s} = \bar{m} + 1, \cdots, m)$ at $x = 0$ and $v_s = \bar{\bar{v}}_s(t)$ $(s = m+1, \cdots, n)$ at $x = L$ on the interval $[0, T]$, we can uniquely determine the initial data $\varphi(x)$ and have the following observability inequality

$$\|\varphi\|_{C^1[0,L]} \leq C\Big(\sum_{\bar{s}=\bar{m}+1}^{m} \|\bar{v}_{\bar{s}}\|_{C^1[0,T]} + \sum_{s=m+1}^{m} \|\bar{\bar{v}}_s\|_{C^1[0,T]} \Big), \tag{1.49}$$

where C is a positive constant.

Remark 1.7. In the case of two-sided observation, the number of boundary observed values is equal to n, the number of all the eigenvalues.

Remark 1.8. In the case of one-sided observation, the number of boundary observed values reduces to the maximum value between the number of positive eigenvalues and the number of negative eigenvalues, and the observation should be taken on the side with less boundary conditions, however, the observability time should be enlarged.

In particular, when the number of positive eigenvalues is equal to the number of negative eigenvalues, the boundary observation can be taken on each side.

Remark 1.9. In the case of two-sided observation with less observed values, both the number of boundary observed values and the observability time are as in the case of one-sided observation, however, one needs all the observed values on the side with more boundary conditions and a part of observed values on the side with less boundary conditions.

Remark 1.10. The estimate on the exact observability time in Theorems 1.4–1.6 is sharp.

1.3 "Duality" Between Controllability and Observability

It is well-known that the exact controllability on $[0, T]$ for the system

$$\frac{dX}{dt} = AX + Bu \tag{1.1}$$

is equivalent to the exact observability on $[0, T]$ for the adjoint system

$$\frac{dZ}{dt} = -A^T Z \tag{1.50}$$

and

$$Y = B^T Z \tag{1.51}$$

(cf [77]).

In the case of hyperbolic PDEs, for the wave equation

$$u_{tt} - \triangle u = 0, \tag{1.6}$$

there is still a duality between controllability and observability. The HUM (Hilbert Uniqueness Method) suggested by J.-L. Lions is to first establish the observability inequality and then to get the controllability via the duality (see [75–76]).

The duality between controllability and observability is only valid in the linear case, but not in the nonlinear case (nonlinear equations or nonlinear boundary conditions). However, comparing Theorems 1.1–1.3 and Theorems 1.4–1.6, it is easy to see that there is still an **implicit duality** between controllability and observability in the quasilinear situation (see Chapter 4).

1.4 Exact Boundary Controllability and Exact Boundary Observability for 1-D Quasilinear Wave Equations

Consider the following quasilinear wave equation

$$u_{tt} - (K(u, u_x))_x = F(u, u_x, u_t), \tag{1.52}$$

where

$$K_v(u, v) > 0 \tag{1.53}$$

and

$$F(0, 0, 0) = 0. \tag{1.54}$$

We prescribe the initial condition

$$t = 0 : \ u = \varphi(x), \ u_t = \psi(x), \quad 0 \le x \le L, \tag{1.55}$$

anyone of the following physically meaningful boundary conditions:

$$x = 0 : \ u = h(t) \quad \text{(Dirichlet type)}, \tag{1.56_1}$$

$$x = 0 : \ u_x = h(t) \quad \text{(Neumann type)}, \tag{1.56_2}$$

$$x = 0: \ u_x - \alpha u = h(t) \quad \text{(Third type)}, \tag{1.56$_3$}$$

$$x = 0: \ u_x - \beta u_t = h(t) \quad \text{(Dissipative type)} \tag{1.56$_4$}$$

(α, β are positive numbers) and a similar boundary condition on $x = L$.

A similar method can be used to get the following results, see Chapter 5 and Chapter 6 (cf. [70–71], [50], also see [43], [45], [48]).

If

$$T > \frac{L}{\sqrt{K_v(0,0)}}, \tag{1.57}$$

then both the **two-sided local exact boundary controllability** and the **two-sided local exact boundary observability** can be realized on the time interval $[0, T]$.

If

$$T > \frac{2L}{\sqrt{K_v(0,0)}}, \tag{1.58}$$

then both the **one-sided local exact boundary controllability** and the **one-sided local exact boundary observability** can be realized on the time interval $[0, T]$.

1.5 Exact Boundary Controllability and Exact Boundary Observability of Unsteady Flows in a Tree-Like Network of Open Canals

A network of canals is called to have a tree-like configuration, if any two nodes in the network can be connected by a unique path of canals. In other words, a tree-like network is a connected network without loop (Figure 1.5).

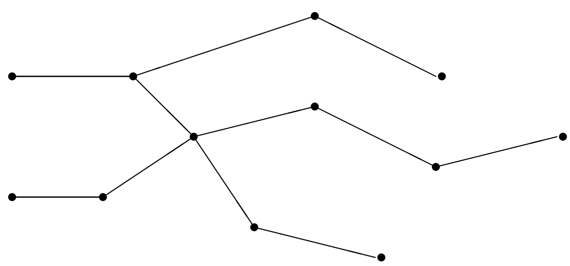

Figure 1.5

The unsteady flow in each canal is described by Saint-Venant system which is a quasilinear hyperbolic system [78]. At each joint point of several canals, there are suitable interface conditions (cf. [40]).

One can choose suitable controls or observed values on certain simple nodes and multiple nodes, such that the corresponding exact boundary controllability or exact boundary observability can be realized respectively in a neighbourhood of a subcritical equilibrium, see Chapter 7 and Chapter 8 (cf., [20–21], [46], [42], [44], [57–58] and [9]).

1.6 Nonautonomous Hyperbolic Systems

For nonautonomous hyperbolic systems

$$\frac{\partial u}{\partial t} + A(t,x,u)\frac{\partial u}{\partial x} = F(t,x,u), \tag{1.59}$$

both the controllability and the observability should depend on the initial time $t = t_0$, and there are various possibilities with delicate behaviors (see Chapter 9).

As an example, we consider the following nonautonomous linear hyperbolic system

$$\begin{cases} \dfrac{\partial r}{\partial t} - f'(t)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + f'(t)\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{1.60}$$

where

$$f'(t) > 0, \quad \forall t \in \mathbb{R}. \tag{1.61}$$

Boundary conditions are given by

$$x = 0: \quad r + s = h(t), \tag{1.62}$$

$$x = L: \quad r - s = g(t). \tag{1.63}$$

The initial condition is

$$t = t_0: \quad (r,s) = (r_0(x), s_0(x)), \quad 0 \le x \le L, \tag{1.64}$$

while, the final condition is

$$t = t_0 + T: \quad (r,s) = (r_T(x), s_T(x)), \quad 0 \le x \le L. \tag{1.65}$$

Setting

$$\bar{t} = f(t), \tag{1.66}$$

system (1.60) reduces to the autonomous hyperbolic system with constant coefficients:

$$
\begin{cases}
\dfrac{\partial r}{\partial \bar{t}} - \dfrac{\partial r}{\partial x} = 0, \\
\dfrac{\partial s}{\partial \bar{t}} + \dfrac{\partial s}{\partial x} = 0.
\end{cases} \tag{1.67}
$$

By Theorem 1.1, there is the two-sided exact boundary controllability on the interval $[t_0, t_0 + T]$, if and only if there is no intersection between the maximum determinate domains of the corresponding forward and backward Cauchy problems for system (1.60) with the initial data (1.64) and the final data (1.65) respectively (Figure 1.6). Then, we have (cf. [66])

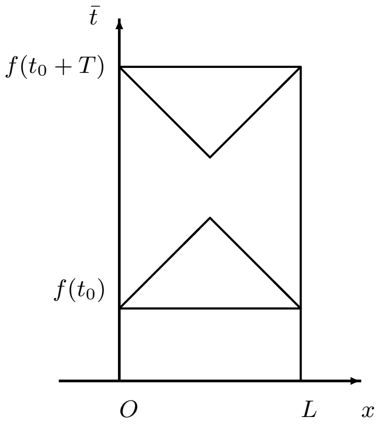

Figure 1.6

Proposition 1.1. For system (1.60) with (1.62)–(1.63), there is the two-sided exact boundary controllability on the interval $[t_0, t_0 + T]$, if and only if

$$
f(t_0 + T) - f(t_0) > L. \tag{1.68}
$$

Therefore, in this situation there are three possibilities:

1. For any given $t_0 \in \mathbb{R}$, we always have the two-sided exact boundary controllability on the interval $[t_0, t_0 + T]$ with

$$
T > T(t_0) \overset{\text{def}}{=} f^{-1}(f(t_0) + L) - t_0. \tag{1.69}
$$

In some special situations, for any given $t_0 \in \mathbb{R}$, the exact controllability time $T > T_0$ can be taken to be independent of t_0.

2. We have the two-sided exact boundary controllability only for a part of t_0 and there is no two-sided exact boundary controllability for the other part of t_0.

3. For any given $t_0 \in \mathbb{R}$, there is no two-sided exact boundary controllability on any finite time interval.

Thus, for the general nonautonomous hyperbolic system (1.59), in the case of two-sided control, the original condition (1.19) given in Theorem 1.1 should be replaced by the following condition (cf. [83]): There exists $T > 0$ such that

$$\int_{t_0}^{t_0+T} \min_{i=1,\cdots,n} \inf_{0\le x\le L} |\lambda_i(t,x,0)|dt > L. \tag{1.70}$$

Similar results hold in the case of one-sided control and in the case of two-sided control with less controls.

The exact boundary observability for nonautonomous hyperbolic systems can be discussed in a similar manner (cf. [27]).

Similar results can be obtained for the following more general 1-D quasilinear wave equation

$$u_{tt} - c^2(t,x,u,u_x,u_t)u_{xx} = f(t,x,u,u_x,u_t) + g(t,x), \tag{1.71}$$

where

$$c(t,x,u,u_x,u_t) > 0 \tag{1.72}$$

and

$$f(t,x,0,0,0) \equiv 0 \tag{1.73}$$

(cf. [28], [84]).

1.7 Notes on the One-Sided Exact Boundary Controllability and Observability

Some notes for the exceptional case are given on the one-sided exact boundary controllability and observability in Chapter 10 and Chapter 11 respectively (cf. [63–65]).

Remark 1.11. The reader may refer to [52] for a similar statement of this Introduction.